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A unified presentation of the generating functions of the generalized Bernoulli, Euler and Genocchi polynomials

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ABSTRACT

The goal of this paper is to unify and extend the generating functions of the generalized Bernoulli polynomials, the generalized Euler polynomials and the generalized Genocchi polynomials associated with the positive real parameters a and b and the complex parameter β. By using this generating function, we derive recurrence relations and other properties for these polynomials. By applying the Mellin transformation to the generating function of the unification of Bernoulli, Euler and Genocchi polynomials, we construct a unification of the zeta functions. Furthermore, we give many properties and applications involving the functions and polynomials investigated in this paper.

1. Introduction, definitions and preliminaries

The definition (1.1) provides us with a generalization and unification of the Bernoulli, Euler and Genocchi polynomials and also of the Apostol–Bernoulli, Apostol–Euler and Apostol–Genocchi polynomials, which were considered in many earlier investigations by (among others) Srivastava et al. [1–6], Karande and Thakare [7], Ozden et al. [8–10], Kim et al. [11], Luo [12], and Simsek [13–15].

Definition 1 (*See, for details, Ozden [9]*). A unification $\mathcal{Y}_{n,\beta}(x;k,a,b)$ of the Bernoulli, Euler and Genocchi polynomials is given by means of the following generating function:

$$\mathfrak{f}_{a,b}(x;t;k,\beta) := \frac{2^{1-k}t^k e^{xt}}{\beta^b e^t - a^b} = \sum_{n=0}^{\infty} \mathcal{Y}_{n,\beta}(x;k,a,b)\,\frac{t^n}{n!} \qquad \left(\left|t + b\log\left(\frac{\beta}{a}\right)\right| < 2\pi;\ x \in \mathbb{R}\right)$$
$$\left(k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}\ (\mathbb{N} := \{1,2,3,\ldots\});\ a,b \in \mathbb{R}^+;\ \beta \in \mathbb{C}\right), \tag{1.1}$$

where, as usual, $\mathbb{R}^+$ and $\mathbb{C}$ denote the sets of *positive* real numbers and *complex* numbers, respectively, $\mathbb{R}$ being the set of *real* numbers.

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Observe that, if we set $x = 0$ in the generating function (1.1), then we obtain the corresponding unification and generalization of the generating functions of not only the Bernoulli and Euler numbers, but also the Genocchi numbers. We thus have

$$\mathcal{Y}_{n,\beta}(0; k, a, b) =: \mathcal{Y}_{n,\beta}(k, a, b). \tag{1.2}$$

Remark 1. In one of the aforementioned investigations, Srivastava et al. [5] considered the following generalization of the Bernoulli and related polynomials:

$$\left(\frac{t}{\lambda b^t - a^t}\right)^{\alpha} \cdot c^{xt} = \sum_{n=0}^{\infty} \mathfrak{B}_n^{(\alpha)}(x; \lambda; a, b, c)\frac{t^n}{n!}$$
$$\left(\left|t \log\left(\frac{b}{a}\right) + \log\lambda\right| < 2\pi;\ \alpha \in \mathbb{C};\ a, b, c \in \mathbb{R}^+\ (a \neq b);\ x \in \mathbb{R};\ 1^{\alpha} := 1\right). \tag{1.3}$$

Thus, by comparing the generating functions (1.1) and (1.3), we derive the following relationship between the families of the polynomials $\mathcal{Y}_{n,\beta}(x; k, a, b)$ and $\mathfrak{B}_n^{(\alpha)}(x; \lambda; a, b, c)$:

$$\mathcal{Y}_{n,\beta}(x; 1, a, b) = a^{-b}\, \mathfrak{B}_n^{(\alpha)}\left(x; \frac{\beta^b}{a^b}; 1, \mathrm{e}, \mathrm{e}\right). \tag{1.4}$$

Remark 2. The generating function in (1.1) is also related to some special polynomials. For example, if we set $\beta = b = 1$ in the generating function (1.1), we arrive at a unification of the Bernoulli, Euler and Genocchi polynomials, which was defined by Karande and Thakare [7] by means of the following generating function:

$$\frac{2^{1-k}t^k\mathrm{e}^{xt}}{\mathrm{e}^t - a} = \sum_{n=0}^{\infty} D_n(x; a, k)\frac{t^n}{n!} \quad \left(|t| < 2\pi;\ x \in \mathbb{R};\ k \in \mathbb{N}_0;\ a \in \mathbb{R}^+\right). \tag{1.5}$$

Remark 3. If we set $k = a = b = 1$ in the generating function (1.1), we get a special case of the generalized Bernoulli polynomials $\mathcal{Y}_{n,\beta}(x; k, a, b)$, that is, the so-called Apostol–Bernoulli polynomials $\mathcal{B}_n(x, \beta)$ generated by

$$\frac{t\mathrm{e}^{xt}}{\beta\mathrm{e}^t - 1} = \sum_{n=0}^{\infty} \mathcal{B}_n(x, \beta)\frac{t^n}{n!} \quad (|t + \log\beta| < 2\pi;\ x \in \mathbb{R}). \tag{1.6}$$

The polynomials $\mathcal{B}_n(x, \beta)$ were investigated by Apostol [16] and, more recently, by Srivastava [4]. For further information and other details, see the recent works [8,1,7,17,11,12,2,9,10,13–15,3–6].

Remark 4. By setting $k + 1 = -a = b = 1$ in the generating function (1.1), we are led to the Apostol–Euler polynomials $\mathcal{E}_n(x, \beta)$, which are defined by means of the following generating function (see, for example, [17,2,10,5]; see also the references cited in each of these earlier works):

$$\frac{2\mathrm{e}^{xt}}{\beta\mathrm{e}^t + 1} = \sum_{n=0}^{\infty} \mathcal{E}_n(x, \beta)\frac{t^n}{n!} \quad (|t + \log\beta| < \pi;\ x \in \mathbb{R}). \tag{1.7}$$

Remark 5. Upon substituting $k = -a = b = 1$ into the generating function (1.1), we have the Apostol–Genocchi polynomials $\mathcal{G}_n(x, \beta)$, which are defined by the following generating function [12,10,5]:

$$\frac{2t\mathrm{e}^{xt}}{\beta\mathrm{e}^t + 1} = \sum_{n=0}^{\infty} \mathcal{G}_n(x, \beta)\frac{t^n}{n!} \quad (|t + \log\beta| < \pi;\ x \in \mathbb{R}), \tag{1.8}$$

so that, obviously,

$$\mathcal{Y}_{n,\beta}(x; 1, -1, 1) = \frac{1}{2}\mathcal{G}_n(x, \beta). \tag{1.9}$$

Remark 6. If we set

$$x = 0, \qquad \beta = \xi q^h \quad \left(\xi^r = 1 \quad (r \in \mathbb{N};\ \xi \neq 1)\right), \quad q \in \mathbb{C} \quad (|q| < 1) \quad \text{and} \quad k = a = b = 1$$

in the generating function (1.1), we arrive at the relationship:

$$\mathcal{Y}_{n,\xi q^h}(1,1,1) = B_{n,\xi}^{(h)}(q) - (\log q^h)\mathcal{B}_n(\xi q^h), \tag{1.10}$$

where $B_{n,\xi}^{(h)}(q)$ denotes twisted (h, q)-extension of the twisted Bernoulli numbers which are defined by the following generating function:

$$\frac{t+\log q^h}{\xi q^h e^t - 1} = \sum_{n=0}^{\infty} B_{n,\xi}^{(h)}(q)\,\frac{t^n}{n!} \quad (|t + h\log q| < 2\pi;\ h \in \mathbb{N}) \tag{1.11}$$

and $\mathcal{B}_n(\xi q^h)$ denotes Apostol–Bernoulli numbers (cf., e.g., [15,18]).

Remark 7. By putting

$$\beta = q^h \quad (q \in \mathbb{C};\ |q| < 1;\ h \in \mathbb{N}), \quad k = 0, a = -1 \quad \text{and} \quad b = h \in \mathbb{N}$$

in the generating function (1.1), we obtain the relationship:

$$\mathcal{Y}_{n,q^h}(0,-1,h) = E_n^{(h)}(q) \tag{1.12}$$

with the (h, q)-Euler numbers $E_n^{(h)}(q)$ defined by the following generating function (cf. [10]):

$$\frac{t+\log q^h}{q^h e^t + 1} = \sum_{n=0}^{\infty} E_n^{(h)}(q)\frac{t^n}{n!} \quad (|t + h\log q| < \pi). \tag{1.13}$$

Remark 8. By setting $\beta = k = b = a = 1$ in the generating function (1.1), we are led to the relationship:

$$\mathcal{Y}_{n,1}(x;1,1,1) = B_n(x) \tag{1.14}$$

with the classical Bernoulli polynomials $B_n(x)$ defined by the following generating function (see, for details, [19, p. 59 *et seq.*]; see also [20]):

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\,\frac{t^n}{n!} \quad (|t| < 2\pi;\ x \in \mathbb{R}). \tag{1.15}$$

Remark 9. If we set $\beta = k + 1 = -a = b = 1$ in the generating function (1.1), we get the relationship:

$$\mathcal{Y}_{n,1}(x;0,-1,1) = E_n(x) \tag{1.16}$$

with the classical Euler polynomials $E_n(x)$ defined by the following generating function (see, for details, [19, p. 63 *et seq.*]; see also [20]):

$$\frac{2e^{xt}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x)\,\frac{t^n}{n!} \quad (|t| < \pi;\ x \in \mathbb{R}). \tag{1.17}$$

Motivated essentially by the interesting connections which are depicted in Remarks 1–9 above, we aim in this paper first at investigating some recurrence relations and other potentially useful properties of the generalized polynomials $\mathcal{Y}_{n,\beta}(x;k,a,b)$ and the generalized numbers $\mathcal{Y}_{n,\beta}(k,a,b)$ defined by (1.1) and (1.2), respectively. We then consider yet another unification of the generating functions of the generalized Bernoulli, Euler and Genocchi polynomials (see Definition 2) by making use of a Dirichlet character χ of conductor $f \in \mathbb{N}$. Finally, in Section 4, we apply the Mellin transformation to the generating function of the polynomials $\mathcal{Y}_{n,\beta}(x;k,a,b)$ in order to present a unification of the families of zeta functions, which provides an interpolation of the polynomials $\mathcal{Y}_{n,\beta}(x;k,a,b)$ for *negative* integer values of n.

2. Properties of the numbers $\mathcal{Y}_{n,\beta}(k,a,b)$ and the polynomials $\mathcal{Y}_{n,\beta}(x;k,a,b)$

The numbers $\mathcal{Y}_{n,\beta}(k,a,b)$ are related rather closely to the Frobenius–Euler numbers $H_n(u)$ which are defined by the following generating function:

$$\frac{1-u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u)\,\frac{t^n}{n!}. \tag{2.1}$$

By using the generating function (1.1), followed by some elementary calculations, we easily obtain Theorem 1.

Theorem 1. *The following relationship holds true between the numbers $\mathcal{Y}_{n,\beta}(k,a,b)$ and the Frobenius–Euler numbers $H_n(u)$ defined by (2.1):*

$$\mathcal{Y}_{n,\beta}(k,a,b)=\frac{k!}{\beta^b-a^b}\binom{n}{k}H_{n-k}\left(\frac{a^b}{\beta^b}\right)\quad\left(\beta^b\neq a^b\right). \tag{2.2}$$

By differentiating both sides of the generating function (1.1) with respect to x, we easily arrive at the following result.

Theorem 2. *The following derivative formula holds true for the polynomials $\mathcal{Y}_{n,\beta}(x;k,a,b)$:*

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\mathcal{Y}_{n,\beta}(x;k,a,b)\right\}=n\mathcal{Y}_{n-1,\beta}(x;k,a,b). \tag{2.3}$$

Carlitz [21] presented a generalization of the Raabe-type multiplication formulas for the Bernoulli and Euler polynomials. Subsequently, Karande and Thakare [7] extended these formulas to hold true for the polynomials $D_n(x;a,k)$ generated by (1.5). The multiplication formula for the polynomials $\mathcal{Y}_{n,\beta}(x;k,a,b)$, which is asserted by Theorem 3, provides a modification and generalization of the Karande–Thakare result [7].

Theorem 3. *The following multiplication formula holds true for the polynomials $\mathcal{Y}_{n,\beta}(x;k,a,b)$:*

$$a^{nb(m-1)}m^{v-k}\sum_{j=0}^{m-1}\left(\frac{\beta}{a}\right)^{bjn}\mathcal{Y}_{v,\beta^m}\left(\frac{x}{m}+\frac{nj}{m};k,a^m,b\right)=a^{mb(n-1)}n^{v-k}\sum_{l=0}^{n-1}\left(\frac{\beta}{a}\right)^{blm}\mathcal{Y}_{v,\beta^n}\left(\frac{x}{n}+\frac{ml}{n};k,a^n,b\right). \tag{2.4}$$

Proof. Our demonstration of the multiplication formula (2.4), based upon the generating function (1.1), is much akin to the method used earlier by Carlitz [21] and, subsequently, by Karande and Thakare [7]. Indeed, by applying the generating function (1.1), we find from the summation identity:

$$\sum_{v=0}^{\infty}\frac{(mt)^v}{v!}\sum_{j=0}^{m-1}\left(\frac{\beta}{a}\right)^{bjn}\mathcal{Y}_{v,\beta^m}\left(\frac{x}{m}+\frac{nj}{m};k,a^m,b\right)=\sum_{j=0}^{m-1}\left(\frac{\beta}{a}\right)^{bjn}\sum_{v=0}^{\infty}\mathcal{Y}_{v,\beta^m}\left(\frac{x}{m}+\frac{nj}{m};k,a^m,b\right)\frac{(mt)^v}{v!}$$

that

$$\begin{aligned}\sum_{v=0}^{\infty}\frac{(mt)^v}{v!}\sum_{j=0}^{m-1}\left(\frac{\beta}{a}\right)^{bjn}\mathcal{Y}_{v,\beta^m}\left(\frac{x}{m}+\frac{nj}{m};k,a^m,b\right)&=\sum_{j=0}^{m-1}\left(\frac{\beta}{a}\right)^{bjn}\left(\frac{2\left(\frac{mt}{2}\right)^k}{\beta^{mb}\mathrm{e}^{mt}-a^{mb}}\,\mathrm{e}^{(x+nj)t}\right)\\&=\left(\frac{2\left(\frac{mt}{2}\right)^k\,\mathrm{e}^{xt}}{\beta^{mb}\mathrm{e}^{mt}-a^{mb}}\right)\sum_{j=0}^{m-1}\left(\frac{\beta}{a}\right)^{bjn}\cdot\mathrm{e}^{njt}\\&=2\left(\frac{mt}{2}\right)^k\mathrm{e}^{xt}\cdot\left(\frac{\beta^{mnb}\mathrm{e}^{mnt}-a^{mnb}}{\left(\beta^{mb}\mathrm{e}^{mt}-a^{mb}\right)\left(\beta^{nb}\mathrm{e}^{nt}-a^{nb}\right)}\right).\end{aligned} \tag{2.5}$$

In precisely the same manner as detailed above (or, alternatively, by observing the symmetry in m and n), we obtain

$$\sum_{v=0}^{\infty}\frac{(nt)^v}{v!}\sum_{j=0}^{n-1}\left(\frac{\beta}{a}\right)^{bjm}\mathcal{Y}_{v,\beta^n}\left(\frac{x}{n}+\frac{mj}{n};k,a^n,b\right)=2\left(\frac{nt}{2}\right)^k\mathrm{e}^{xt}\cdot\left(\frac{\beta^{mnb}\mathrm{e}^{mnt}-a^{mnb}}{\left(\beta^{mb}\mathrm{e}^{mt}-a^{mb}\right)\left(\beta^{nb}\mathrm{e}^{nt}-a^{nb}\right)}\right). \tag{2.6}$$

The assertion (2.4) of Theorem 3 would now follow by comparing these last two results (2.5) and (2.6). □

Remark 10. In its special case when $\beta=b=1$, Theorem 3 yields (cf. [7])

$$a^{n(m-1)}m^{v-k}\sum_{j=0}^{m-1}\left(\frac{1}{a}\right)^{jn}D_v\left(\frac{x}{m}+\frac{nj}{m};k,a^m\right)=a^{m(n-1)}n^{v-k}\sum_{l=0}^{n-1}\left(\frac{1}{a}\right)^{lm}D_v\left(\frac{x}{n}+\frac{lm}{n};k,a^n\right), \tag{2.7}$$

where the polynomials $D_n\,(x;k,a)$ are generated by (1.5).

Remark 11. By setting $m=1$ in Theorem 3, we get Ozden's multiplication formula [9, p. 3, Theorem 3]:

$$\mathcal{Y}_{v,\beta}(x;k,a,b)=a^{b(n-1)}n^{v-k}\sum_{l=0}^{n-1}\left(\frac{\beta}{a}\right)^{bl}\mathcal{Y}_{v,\beta^n}\left(\frac{x+l}{n};k,a^n,b\right). \tag{2.8}$$

Remark 12. Upon substituting $k = m = a = b = 1$ into Theorem 3, we obtain the following multiplication formula for the Apostol–Bernoulli polynomials $\mathcal{B}_n(x, \beta)$ generated by (1.6):

$$\mathcal{B}_\upsilon(x, \beta) = n^{\upsilon-1} \sum_{l=0}^{n-1} \beta^l \, \mathcal{B}_\upsilon\left(\frac{x+l}{n}, \beta^n\right). \tag{2.9}$$

Remark 13. By setting $k + 1 = m = -a = b = 1$, in Theorem 3, we have the following multiplication formula for the Apostol–Euler polynomials $\mathcal{E}_n(x, \beta)$ generated by (1.7):

$$\mathcal{E}_\upsilon(x, \beta) = n^{\upsilon-1} \sum_{l=0}^{n-1} \beta^l \, \mathcal{E}_\upsilon\left(\frac{x+l}{n}, \beta^n\right). \tag{2.10}$$

Remark 14. If we set $k = m = b = -a = 1$ in Theorem 3, we get the following multiplication formula for the Apostol–Genocchi polynomials $\mathcal{G}_n(x, \beta)$ generated by (1.8):

$$\mathcal{G}_\upsilon(x, \beta) = n^{\upsilon-1} \sum_{l=0}^{n-1} \beta^l \, \mathcal{G}_\upsilon\left(\frac{x+l}{n}, \beta^n\right). \tag{2.11}$$

Remark 15. In their *further* special cases when $\beta = 1$, (2.9) to (2.11) would yield Raabe-type multiplication formulas for the classical Bernoulli, Euler and Genocchi polynomials, respectively.

By integrating both sides of the generating function (1.1) with respect to x from $x = 0$ to $x = y$, we obtain

$$\frac{2\left(\frac{t}{2}\right)^k}{\beta^b e^t - a^b}\left(\frac{e^{yt}-1}{t}\right) = \sum_{n=0}^{\infty}\left(\int_0^y \mathcal{Y}_{n,\beta}(x; k, a, b)\mathrm{d}x\right)\frac{t^n}{n!},$$

which, in light of the generating function (1.1) itself, yields

$$\sum_{n=1}^{\infty}\left(\frac{\mathcal{Y}_{n+1,\beta}(y; k, a, b) - \mathcal{Y}_{n+1,\beta}(k, a, b)}{n+1}\right)\frac{t^n}{n!} = \int_0^y \mathcal{Y}_{0,\beta}(x; k, a, b)\mathrm{d}x + \sum_{n=1}^{\infty}\left(\int_0^y \mathcal{Y}_{n,\beta}(x; k, a, b)\mathrm{d}x\right)\frac{t^n}{n!}. \tag{2.12}$$

Thus, by equating the coefficients of $\frac{t^n}{n!}$ on both sides of Eq. (2.12), we obtain the following result.

Theorem 4. *The following integral formula holds true:*

$$\int_0^y \mathcal{Y}_{n,\beta}(x; k, a, b)\mathrm{d}x = \begin{cases} \dfrac{\mathcal{Y}_{n+1,\beta}(y; k, a, b) - \mathcal{Y}_{n+1,\beta}(k, a, b)}{n+1} & (n \in \mathbb{N}) \\ 0 & (n = 0). \end{cases} \tag{2.13}$$

3. Unification of the generating functions of the generalized Bernoulli, Euler and Genocchi polynomials and numbers by using a Dirichlet character

In this section, in terms of a Dirichlet character χ of conductor $f \in \mathbb{N}$, we extend and investigate the generating functions of the generalized Bernoulli, Euler and Genocchi numbers and the generalized Bernoulli, Euler and Genocchi polynomials with parameters a, b, β and k (see Definition 1). Such χ-extended polynomials and χ-extended numbers are potentially useful in many areas of Mathematics and Mathematical Physics.

Definition 2. Let χ be a Dirichlet character of conductor $f \in \mathbb{N}$. Let $k \in \mathbb{N}$, $a, b \in \mathbb{R}^+$ and $\beta \in \mathbb{C}$. Then the proposed extensions of the generating functions of the generalized Bernoulli, Euler and Genocchi numbers $\mathcal{Y}_{n,\beta}(k, a, b)$ and the generalized Bernoulli, Euler and Genocchi polynomials $\mathcal{Y}_{n,\beta}(x; k, a, b)$ are given by

$$\mathcal{F}_{\chi,\beta}(t, k, a, b) := 2^{1-k}t^k \sum_{j=1}^{f} \frac{\chi(j)\left(\frac{\beta}{a}\right)^j e^{jt}}{\beta^{bf}e^{ft} - a^{bf}} = \sum_{n=0}^{\infty} \mathcal{Y}_{n,\chi,\beta}(k, a, b)\,\frac{t^n}{n!}$$

$$\left(\left|t + b\log\left(\frac{\beta}{a}\right)\right| < 2\pi;\ f, k \in \mathbb{N};\ a, b \in \mathbb{R}^+;\ \beta \in \mathbb{C}\right) \tag{3.1}$$

and

$$\mathfrak{H}_{\chi,\beta}(x;t,k,a,b) := \mathcal{F}_{\chi,\beta}(t,k,a,b)e^{xt} = \sum_{n=0}^{\infty} \mathcal{Y}_{n,\chi,\beta}(x;k,a,b)\,\frac{t^n}{n!}$$
$$\left(\left|t + b\log\left(\frac{\beta}{a}\right)\right| < 2\pi;\; x\in\mathbb{R};\; f,k\in\mathbb{N};\; a,b\in\mathbb{R}^+;\; \beta\in\mathbb{C}\right), \tag{3.2}$$

respectively.

Remark 16. If we set $\chi \equiv 1$ in (3.2), we are led immediately to the generating function (1.1). The corresponding χ-extended versions $\mathcal{B}_{n,\chi}(x,\beta)$ of the Apostol–Bernoulli polynomials, $\mathcal{E}_{n,\chi}(x,\beta)$ of the Apostol–Euler polynomials $\mathcal{E}_{n,\chi}(x,\beta)$ and $\mathcal{G}_{n,\chi}(x,\beta)$ of the Apostol–Genocchi polynomials (cf. [11,2]; see also the references cited in each of these earlier works) are given by

$$\mathcal{B}_{n,\chi}(x,\beta) := \mathcal{Y}_{n,\chi,\beta}(x;1,1,1), \tag{3.3}$$

$$\mathcal{E}_{n,\chi}(x,\beta) := \mathcal{Y}_{n,\beta}(x;0,-1,1) \tag{3.4}$$

and

$$\mathcal{G}_{n,\chi}(x,\beta) := 2\,\mathcal{Y}_{n,\chi,\beta}(x;1,-1,1), \tag{3.5}$$

respectively. Moreover, by *further* setting $\beta = 1$ in (3.3)–(3.5), we can get the following χ-extended versions of the classical Bernoulli, Euler and Genocchi polynomials (see, for example, [19]; see also many of the recent works cited in this paper):

$$B_{n,\chi}(x) := \mathcal{B}_{n,\chi}(x) = \mathcal{Y}_{n,\chi,1}(x;1,1,1), \tag{3.6}$$

$$E_{n,\chi}(x) := \mathcal{Y}_{n,\chi,1}(x;0,-1,1) = \mathcal{E}_{n,\chi}(x) \tag{3.7}$$

and

$$G_{n,\chi}(x) := \mathcal{G}_{n,\chi}(x) := 2\,\mathcal{Y}_{n,\chi,1}(x;1,-1,1), \tag{3.8}$$

respectively.

Next, by applying the generating functions (3.1) and (3.2), we find that

$$\sum_{n=0}^{\infty} \mathcal{Y}_{n,\chi,\beta}(k,a,b)\,\frac{t^n}{n!}\sum_{n=0}^{\infty}\frac{(xt)^n}{n!} = \sum_{n=0}^{\infty} \mathcal{Y}_{n,\chi,\beta}(x;k,a,b)\,\frac{t^n}{n!}. \tag{3.9}$$

Thus, by using the Cauchy product in (3.9) and then equating the coefficients of $\frac{t^n}{n!}$ on both sides of the resulting equation, we obtain the following explicit representation for the polynomials $\mathcal{Y}_{n,\chi,\beta}(x;k,a,b)$ generated by (3.2).

Theorem 5. *Let $a,b\in\mathbb{R}$, $k\in\mathbb{N}$ and $\beta\in\mathbb{C}$. Then*

$$\mathcal{Y}_{n,\chi,\beta}(x;k,a,b) = \sum_{j=0}^{n}\binom{n}{j} x^{n-j}\,\mathcal{Y}_{j,\chi,\beta}(k,a,b). \tag{3.10}$$

4. A unification of the family of the zeta functions

Our main objective in this section is to apply the Mellin transformation to the generating function Eq. (1.1) of the polynomials $\mathcal{Y}_{n,\beta}(x;k,a,b)$ in order to construct a unification of the various members of the family of the zeta functions and to thereby interpolate $\mathcal{Y}_{n,\beta}(x;k,a,b)$ for *negative* integer values of n.

Throughout this section, we assume that

$$\beta\in\mathbb{C}\quad(|\beta|<1)\quad\text{and}\quad s\in\mathbb{C}.$$

Indeed, by applying the Mellin transformation to the generating function (1.1), we have the following integral representation of the unified zeta function $\zeta_\beta(s,x;k,a,b)$:

$$\zeta_\beta(s,x;k,a,b) = \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-k-1}\,\mathfrak{f}_{a,b}(x;-t;k,\beta)\mathrm{d}t \quad \left(\Re(s)>k;\; k\in\mathbb{N}_0\right) \tag{4.1}$$

in terms of the generating function $\mathfrak{f}_{a,b}(x;t;k,\beta)$ defined in (1.1). Thus, by making use of (4.1), we are ready to define a unification $\zeta_\beta(s,x;k,a,b)$ of the Hurwitz-type zeta functions as follows.

Definition 3. For

$$\beta \in \mathbb{C} \quad (|\beta| < 1) \quad \text{and} \quad s \in \mathbb{C},$$

we define a unified zeta function $\zeta_\beta(s, x; k, a, b)$ by

$$\zeta_\beta(s, x; k, a, b) := \left(-\frac{1}{2}\right)^{k-1} \sum_{n=0}^{\infty} \frac{\beta^{bn}}{a^{b(n+1)}\,(n+x)^s} \quad \big(\Re(s) > 1\big). \tag{4.2}$$

Remark 17. By setting $x = 1$ in (4.2), we have a unification of the Riemann-type zeta functions as follows:

$$\zeta_\beta(s; k, a, b) := \left(-\frac{1}{2}\right)^{k-1} \sum_{n=1}^{\infty} \frac{\beta^{b(n-1)}}{a^{bn}\, n^s} \quad \big(\Re(s) > 1\big). \tag{4.3}$$

By applying (4.2) and (4.3), we easily arrive at Theorem 6.

Theorem 6. *Suppose that*

$$\beta \in \mathbb{C} \quad (|\beta| < 1) \quad \textit{and} \quad s \in \mathbb{C}.$$

Then

$$\zeta_\beta(s, x; k, a, b) = \frac{\left(-\frac{1}{2}\right)^{k-1}}{N^s} \sum_{j=1}^{N} \left(\frac{\beta^{j-1}}{a^{j-N}}\right)^b \zeta_{\beta^N}\left(s, \frac{x+j-1}{N}; k, a^N, b\right) \tag{4.4}$$

and

$$\zeta_\beta(s; k, a, b) = \frac{\left(-\frac{1}{2}\right)^{k-1}}{N^s} \sum_{j=1}^{N} \left(\frac{\beta^{j-1}}{a^{j-N}}\right)^b \zeta_{\beta^N}\left(s, \frac{j}{N}; k, a^N, b\right). \tag{4.5}$$

Remark 18. Both the Hurwitz (or generalized) zeta function $\zeta(s, x)$ and the Riemann zeta function $\zeta(s)$ are obvious special cases of the unified zeta function $\zeta_\beta\,(s, x; k, a, b)$ defined by (4.2). We thus have

$$\zeta_1(s, x; 1, 1, 1) = \zeta(s, x) := \sum_{n=0}^{\infty} \frac{1}{(n+x)^s} \quad \big(\Re(s) > 1\big) \tag{4.6}$$

and

$$\zeta_1(s, 1; 1, 1, 1) = \zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \quad \big(\Re(s) > 1\big). \tag{4.7}$$

It is known that each of these zeta functions can be continued meromorphically to the whole complex s-plane with a *simple* pole at $s = 1$ with residue 1 (see, for details, [19,20]). Furthermore, in its special case when $a = b = k = \beta = 1$, (4.5) yields the following well-known identity (see, for example, [4, p. 79, Eq. (2.8)]):

$$\zeta(s) = \frac{1}{N^s} \sum_{j=1}^{N} \zeta\left(s, \frac{j}{N}\right) \tag{4.8}$$

for the Riemann zeta function $\zeta(s)$. More generally, for the Hurwitz (or generalized) zeta function $\zeta(s, x)$, it is known that (see, for example, [4, p. 82, Eq. (3.1)])

$$\zeta(s, x) = \frac{1}{N^s} \sum_{j=1}^{N} \zeta\left(s, \frac{x+j-1}{N}\right), \tag{4.9}$$

which immediately yields (4.8) when $x = 1$. Obviously, this last identity (4.9) would follow as a special case of the assertion (4.4) of Theorem 6 when $a = b = k = \beta = 1$.

The unified zeta function $\zeta_\beta(s, x; k, a, b)$ interpolates the polynomials $\mathcal{Y}_{n,\beta}(x; k, a, b)$ for *negative* integer values of n.

Theorem 7. *Let $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$. Then*

$$\zeta_\beta(1-n, x; k, a, b) = (-1)^k \frac{(n-1)!}{(n+k-1)!} \mathcal{Y}_{n+k-1,\beta}(x; k, a, b). \tag{4.10}$$

Proof. The proof of Theorem 7 would run parallel to that of a known result due to Srivastava et al. [22, p. 254, Theorem 8]. By using the same method of Srivastava et al. (see, for details, [22, p. 254]) and using the Cauchy residue theorem in conjunction with (4.1), we arrive at the desired result (4.10). □

We now give some further properties of the unified zeta function $\zeta_\beta(s, x; k, a, b)$. First of all, upon setting $s = 1-n$ $(n \in \mathbb{N})$ in assertions (4.4) and (4.5) of Theorem 6, if we make use of (4.10), we obtain the corresponding multiplication formulas for the polynomials $\mathcal{Y}_{n+k-1,\beta}(x; k, a, b)$ and the numbers $\mathcal{Y}_{n+k-1,\beta}(k, a, b)$, which are contained in Theorem 8.

Theorem 8. *The following multiplication formulas hold true for the polynomials $\mathcal{Y}_{n+k-1,\beta}(x; k, a, b)$ and the numbers $\mathcal{Y}_{n+k-1,\beta}(k, a, b)$:*

$$\mathcal{Y}_{n+k-1,\beta}(x; k, a, b) = \left(-\frac{1}{2}\right)^{k-1} N^{n-1} \sum_{j=1}^{N} \left(\frac{\beta^{j-1}}{a^{j-N}}\right)^b \cdot \mathcal{Y}_{n+k-1,\beta^N}\left(\frac{x+j-1}{N}; k, a^N, b\right) \tag{4.11}$$

and

$$\mathcal{Y}_{n+k-1,\beta}(k, a, b) = \left(-\frac{1}{2}\right)^{k-1} N^{n-1} \sum_{j=1}^{N} \left(\frac{\beta^{j-1}}{a^{j-N}}\right)^b \mathcal{Y}_{n+k-1,\beta^N}\left(\frac{j}{N}; k, a^N, b\right). \tag{4.12}$$

Remark 19. If we set

$$\beta = \xi q^h \quad \left(\xi^r = 1 \quad (r \in \mathbb{N};\ \xi \neq 1);\ h \in \mathbb{Z}\right), \quad q \in \mathbb{C} \quad (|q| < 1) \quad \text{and} \quad b = a = 1$$

in Theorem 8, we arrive at a known result given by Aygunes and Simsek [23] (see also Ozden [9] for a different proof of Theorem 8 by using generating functions).

Another important property of the unified zeta function $\zeta_\beta(s, x; k, a, b)$ is given by the following proposition involving the familiar Hurwitz–Lerch zeta function $\Phi(z, s, a)$ defined by (cf., e.g., [19, p. 121 *et seq.*])

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \quad \left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ s \in \mathbb{C} \quad \text{when} \quad |z| < 1;\ \Re(s) > 1 \quad \text{when} \quad |z| = 1\right), \tag{4.13}$$

where, as usual,

$$\mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\} \quad (\mathbb{Z}^- := \{-1, -2, -3, \ldots\}).$$

Proposition. *The following relationship holds true between the unified zeta function $\zeta_\beta(s, x; k, a, b)$ and the Hurwitz–Lerch zeta function $\Phi(z, s, a)$:*

$$\zeta_\beta(s, x; k, a, b) := \left(-\frac{1}{2}\right)^{k-1} a^b\, \Phi\left(\frac{\beta^b}{a^b}, s, x\right). \tag{4.14}$$

Remark 20. The Hurwitz–Lerch zeta function $\Phi(z, s, a)$ contains not only the Riemann zeta function $\zeta(s)$ and the Hurwitz (or generalized) zeta function $\zeta(s, a)$ given by

$$\zeta(s) = \Phi(1, s, 1) \quad \text{and} \quad \zeta(s, a) = \Phi(1, s, a) \tag{4.15}$$

and the Lerch Zeta function $\ell_s(\tau)$ given by

$$\ell_s(\tau) := \sum_{n=1}^{\infty} \frac{e^{2n\pi i\tau}}{n^s} = e^{2\pi i\tau}\, \Phi\left(e^{2\pi i\tau}, s, 1\right) \quad (\tau \in \mathbb{R};\ \Re(s) > 1), \tag{4.16}$$

but also such other important functions of *Analytic Number Theory* as the Polylogarithm function $\mathrm{Li}_s(z)$ given by

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s} = z\, \Phi(z, s, 1) \quad \left(s \in \mathbb{C} \quad \text{when} \quad |z| < 1;\ \Re(s) > 1 \quad \text{when} \quad |z| = 1\right) \tag{4.17}$$

and the Lipschitz–Lerch zeta function $\phi(\tau, a, s)$ given by (cf. [19, p. 122, Eq. (2.5) (11)]):

$$\phi(\tau, a, s) := \sum_{n=0}^{\infty} \frac{e^{2n\pi i\tau}}{(n+a)^s} = \Phi\left(e^{2\pi i\tau}, s, a\right) =: L(\tau, s, a)$$
$$\left(a \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \Re(s) > 0 \quad \text{when} \quad \tau \in \mathbb{R} \setminus \mathbb{Z};\ \Re(s) > 1 \quad \text{when} \quad \tau \in \mathbb{Z}\right), \tag{4.18}$$

which was first studied by Rudolf Lipschitz (1832–1903) and Matyáš Lerch (1860–1922) in connection with Dirichlet's famous theorem on primes in arithmetic progressions (cf., e.g., [3,5]).

For the Hurwitz–Lerch zeta function $\Phi(z, s, a)$ defined by (4.13), it is also known that (see, for example, [4, p. 81, Eq. (3.8)])

$$\Phi(z, s, a) = \frac{1}{N^s} \sum_{j=1}^{N} \Phi\left(z^N, s, \frac{a+j-1}{N}\right) z^{j-1}, \tag{4.19}$$

which obviously belongs to the family of such identities as (4.4), (4.5), (4.8) and (4.9).

Remark 21. In recent years, some interesting *multi-parameter* generalizations of the Hurwitz–Lerch zeta function $\Phi(z, s, a)$ were investigated by (for example) Garg et al. [24], Lin et al. [25] and Choi et al. [26]. Moreover, the interested reader should refer also to the works by (among others) Ráducanu and Srivastava [27] and by Gupta et al. [28] for some recent applications of the general Hurwitz–Lerch zeta function $\Phi(z, s, a)$ in *Geometric Function Theory of Complex Analysis* and in *Probability Distribution Theory*, respectively (cf. [3,5]; see also [29,30]).

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